6. S. K. Godunov and G. P. Prokopov, "On the solution of differential equations using curvilinear difference nets," Zh. Vychisl. Mat. Mat. Fiz., 8, 28 (1968).
7. A. F. Emery and W. W. Carson, "An evaluation of the use of the finite-element method in the computation of temperature," Proc. Am. Soc. Mech. Eng., Ser. C. Heat Transfer, 93, 136 (1971).
8. G. K. Malikov, "A numerical method for solving problems of unsteady nonlinear heat conduction for two-dimensional bodies of complex shape," Inzh.-Fiz. Zh., 32, 905 (1977).
9. A. A. Samarskii, Introduction to the Theory of Difference Schemes [in Russian], Nauka, Moscow (1971).
10. P. P. Belinskii, S. K. Godunov, Yu. B. Ivanov, and I. K. Yanenko, "Application of one class of quasiconformal mappings for the construction of difference nets in regions with curvilinear boundaries," Zh. Vychisl. Mat. Mat. Fiz., 15, 1499 (1975).
11. P. F. Fil'chakov, Approximate Methods of Conformal Mappings [in Russian], Naukova Dumka, Kiev (1964).

ANALYTIC SOLUTIONS OF PARABOLIC AND HYPERBOLIC HEAT-TRANSFER EQUATIONS FOR NONLINEAR MEDIA

O. N. Shablovskii

UDC 536.2.01

New classes of analytic solutions are obtained which describe unsteady temperature distributions and take account of the temperature dependence of the thermophysical properties of the material. The concept of a solution of the boundary layer transition type is introduced for the generalized heat-transfer equation.

We consider the nonlinear heat-conduction equation in a one-dimensional plane region

$$c(T)\,T_t = [\lambda(T)\,T_x]_x. \tag{1}$$

We introduce a new function $\xi = \xi(x, t)$ with the following properties:

$$\xi_x = U(T),\quad \xi_t = \lambda T_x,$$
$$U(T) = U_0 + \int_0^T c(T)\,dT,\quad U_0 \equiv \text{const}.$$

We change from the variables (x, t) to new independent variables (ξ, t):

$$d\xi = U(T)\,dx + (\lambda U T_\xi)\,dt,$$
$$D(\xi,\ t)/D(x,\ t) = U \neq 0,$$

so that the initial Eq. (1) takes the form

$$\beta(T)\,T_t = [\lambda(T)\,T_\xi]_\xi,\ \ \beta = cU^{-2},\ \ T = T(\xi,\ t), \tag{2}$$

where the Cartesian coordinate is related to the new variable by the equation

$$x(\xi,\ t) = \int \frac{d\xi}{U(\xi,\ 0)} - \int_0^t \lambda T_\xi dt,\quad U = U[T(\xi,\ t)]. \tag{3}$$

A comparison of Eqs. (1) and (2) shows that to each one-dimensional unsteady temperature distribution in a medium with the thermophysical parameters c(T) and λ(T) there corresponds a certain one-dimensional unsteady temperature distribution in a medium with volumetric heat capacity β(T) and a thermal conductivity

Scientific-Research Institute of Applied Mathematics and Mechanics, Tomsk State University. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 40, No. 3, pp. 510-517, March, 1981. Original article submitted January 29, 1980.

$\lambda(T)$. Consequently, different temperature distributions in different media are equivalent from the point of view of the transformation considered. Thus, it is possible to construct new classes of temperature distributions based on known exact and approximate solutions of the nonlinear heat-conduction equation. The boundaries of the spatial region in which the old solution was obtained are transformed by Eq. (3).

We note, e.g., that if $\beta(T) \equiv \text{const}$

$$c(T) = \beta\left(\beta T - \frac{1}{U_0}\right)^{-2}.$$

We present an example of the solution of the nonlinear hyperbolic heat-transfer equation in a semiinfinite region with a moving boundary. We take the generalized heat-transfer equation [1-3] in the form

$$cT_t + c\gamma T_{tt} + q_v = (\lambda T_x)_x. \tag{4}$$

We consider a temperature interval $T \in (0, T']$ in which the thermophysical properties of the material vary as follows:

$$c(T) = c_0 + c_1 T, \quad \lambda(T) = \lambda_0 + \lambda_1 T, \quad c_i, \ \lambda_i - \text{const}, \ i = 0, \ 1,$$

$$\gamma(T, \ t) = \gamma_0 + \gamma_1(t)\, T, \quad \gamma_0 \equiv \text{const}, \ \gamma \geqslant 0.$$

The internal heat source strength is $q_v = \sum_{i=2}^{5} q_i(t)\, T^i$. In the class of solutions given below we assume that if the value $\gamma = 0$ is not excluded in considering heat transfer in the temperature range under study, $q_2 \neq 0$. If $\gamma > 0$ everywhere, we can take $q_2 = 0$, $q_v = 0$.

The initial and boundary conditions are:

$$t = 0: T(x, \ 0) = T^{(0)}(x), \ T_t(x, \ 0) = T^{(1)}(x); \ 0 \leqslant t < \infty,$$

$$x \to -\infty, \ T \to 0; \ x = x_b(t), \ T = T_b(t), \ -\infty < x \leqslant x_b(t).$$

Equation (4) is satisfied by the following expressions:

$$\xi_x = S(T), \ \lambda T_x = \xi_t + \eta, \ \eta_x = q_v + c\gamma T_{tt}, \ \eta = \eta(x, \ t),$$

$$S \equiv U - U_0 = \int_0^T c(T)\, dT.$$

We introduce new independent variables (ξ, t):

$$d\xi = S dx + (\lambda S T_\xi - \eta)\, dt,$$

$$D(\xi, \ t)/D(x, \ t) = S \neq 0, \quad 0 < T \leqslant T', \tag{5}$$

$$x(\xi, \ t) = \int \frac{d\xi}{S(\xi, \ 0)} - \int_0^t \frac{\lambda S T_\xi - \eta}{S}\, dt$$

and instead of (4) we obtain for $T(\xi, t)$ and $\eta(\xi, T)$:

$$(c_0 + c_1 T)(T_t - \eta T_\xi) + S\eta_\xi = S^2(\lambda T_{\xi\xi} + \lambda_1 T_\xi^2), \tag{6}$$

$$S\eta_\xi = q_v + c\gamma\left[T_{tt} + 3\lambda S T_\xi T_{\xi t} - 2\eta T_{\xi t} - \eta_t T_\xi + \frac{d(\lambda S)}{dT} T_\xi^2 T_t\right]. \tag{7}$$

Equation (6) is hyperbolic for $\gamma > 0$ if $S\eta_\xi$ is replaced by (7).

We seek the solutions of (6) and (7) in the form

$$T(\xi, \ t) = \sum_{n=1}^{\infty} T_n(t)\, \xi^n, \quad \eta(\xi, \ t) = \sum_{n=1}^{\infty} \eta_{n+1}(t)\, \xi^{n+1}, \ \xi \in (0, \ \xi_b]. \tag{8}$$

Substitution of these series into the equations gives

$$\sum_{n=1}^{\infty}\Big\{\Big[c_0\dot{T}_{n+1}+\sum_{k=1}^{n}c_0T_k\eta_{n+2-k}(n+2-2k)-l_2k(k+1)T_{k+1}A_{n+2-k}-$$

$$-\lambda_1c_0^2A_{k+1}E_{n-k}\Big]\xi^{n+1}+\Big[\sum_{k=1}^{n}c_1T_k\dot{T}_{n+2-k}-c_1\eta_{k+1}\times$$

$$\times\Big(\sum_{i=1}^{n-k+1}T_i(n-k+2-i)T_{n-k+2-i}\Big)+\frac{c_1}{2}(k+1)\eta_{k+1}A_{n+2-k}-$$

$$-l_3k(k+1)T_{k+1}C_{n+3-k}-\lambda_1c_0c_1C_{k+2}E_{n-k}\Big]\xi^{n+2}+$$

$$+\Big[\sum_{k=1}^{n}(-l_4)k(k+1)T_{k+1}B_{n+4-k}-\lambda_1\frac{c_1^2}{4}B_{k+3}E_{n-k}\Big]\xi^{n+3}-$$

$$-l_5\Big[\sum_{k=1}^{n}k(k+1)T_kD_{n+5-k}\Big]\xi^{n+4}\Big\}=0;$$

$$\sum_{n=1}^{\infty}\Big\{\Big[c_0\gamma_0\ddot{T}_{n+1}+q_2A_{n+1}+\sum_{k=1}^{n}3\beta_1T_kF_{n+1-k}-c_0T_k(n+2-k)\eta_{n+2-k}-$$

$$-c_0\gamma_0kT_k\dot{\eta}_{n+2-k}+\alpha_0\dot{T}_{k+1}E_{n-k}\Big]\xi^{n+1}+\Big[q_3C_{n+2}+\sum_{k=1}^{n}(c_0\gamma_1+$$

$$+\gamma_0c_1)T_k\ddot{T}_{n+2-k}-\frac{c_1}{2}(k+1)\eta_{k+1}A_{n+2-k}+3\beta_2A_{k+1}F_{n+1-k}-$$

$$-2c_0\gamma_0H_{n+2}-(c_0\gamma_1+\gamma_0c_1)T_kG_{n+2-k}+\alpha_1T_kK_{n+2-k}\Big]\xi^{n+2}+$$

$$+\Big[q_4B_{n+3}+\sum_{k=1}^{n}c_1\gamma_1A_{k+1}\ddot{T}_{n+2-k}+3\beta_3C_{k+2}F_{n+1-k}-$$

$$-2(c_0\gamma_1+\gamma_0c_1)T_kH_{n+3-k}-c_1\gamma_1A_{k+1}G_{n+2-k}+\alpha_2A_{k+1}K_{n+2-k}\Big]\xi^{n+3}+$$

$$+\Big[q_5D_{n+4}+\sum_{k=1}^{n}3\beta_4B_{k+3}F_{n+1-k}-2c_1\gamma_1A_{k+1}H_{n+3-k}+\alpha_3C_{k+2}K_{n+2-k}\Big]\xi^{n+4}+$$

$$+\Big[\sum_{k=1}^{n}3\beta_5D_{k+4}F_{n+1-k}+\alpha_4B_{k+3}K_{n+2-k}\Big]\xi^{n+5}\Big\}=0;$$

$$\alpha_0=\lambda_0c_0^2\gamma_0,\quad \alpha_1=c_0\gamma_0(\lambda_0c_1+2\lambda_1c_0)+\lambda_0c_0(c_0\gamma_1+\gamma_0c_1),$$

$$\alpha_2=\frac{3}{2}\lambda_1c_1c_0\gamma_0+(c_0\gamma_1+\gamma_0c_1)(\lambda_0c_1+2\lambda_1c_0)+c_1\gamma_1\lambda_0c_0,$$

$$\alpha_3=\frac{3}{2}\lambda_1c_1(c_0\gamma_1+\gamma_0c_1)+c_1\gamma_1(\lambda_0c_1+2\lambda_1c_0),\quad \alpha_4=\frac{3}{2}\lambda_1c_1^2\gamma_1,$$

$$\beta_1=\alpha_0,\quad \beta_2=c_0\gamma_0\Big(\frac{\lambda_0c_1}{2}+\lambda_1c_0\Big)+\lambda_0c_0(c_0\gamma_1+\gamma_0c_1),$$

$$\beta_3=\Big(\frac{\lambda_0c_1}{2}+\lambda_1c_0\Big)(c_0\gamma_1+\gamma_0c_1)+\lambda_0c_0c_1\gamma_1+\frac{\lambda_1c_1}{2}c_0\gamma_0,$$

$$\beta_4=\frac{\lambda_1c_1}{2}(c_0\gamma_1+\gamma_0c_1)+c_1\gamma_1\Big(\frac{\lambda_0c_1}{2}+\lambda_1c_0\Big),\quad \beta_5=\frac{\lambda_1}{2}c_1^2\gamma_1,$$

$$l_2=\lambda_0c_0^2,\quad l_3=\lambda_0c_0c_1+\lambda_1c_0^2,\quad l_4=\frac{\lambda_0}{4}c_1^2+\lambda_1c_1c_0,\quad l_5=\frac{\lambda_1}{4}c_1^2,$$

$$A_{n+1}=\sum_{k=1}^{n}T_kT_{n-k+1},\qquad C_{n+2}=\sum_{k=1}^{n}T_kA_{n+2-k}.$$

$$B_{n+3}=\sum_{k=1}^{n}A_{k+1}A_{n-k+2},\quad D_{n+4}=\sum_{k=1}^{n}T_kB_{n+4-k},$$

$$E_{n-1}=\sum_{k=1}^{n}kT_k(n+1-k)T_{n+1-k},\quad F_n=\sum_{k=}^{n}k(n+2-k)T_k\dot{T}_{n+2-k},$$

$$G_{n+1}=\sum_{k=1}^{n} kT_k\dot{\eta}_{n+2-k},\quad H_{n+2}=\sum_{k=1}^{n}(k+1)\dot{T}_{k+1}\eta_{n+2-k},$$

$$K_{n+1}=\sum_{k=1}^{n}\dot{T}_{k+1}E_{n-k}.$$

Hence we find that the expansion coefficients in (8) must be calculated from the recurrence relations

$$T_1\equiv\text{const}\neq 0,\ \dot{T}_{n+1}-n(n+1)\theta_1 T_{n+1}+nT_1\eta_{n+1}+\Phi_{n+1}(t)=0,$$
$$\theta_1=\lambda_0 c_0 T_1^2, \tag{9}$$
$$\gamma_0\ddot{T}_{n+1}+(3n+4)\gamma_0\theta_1\dot{T}_{n+1}-\gamma_0 T_1\dot{\eta}_{n+1}-(n+1)T_1\eta_{n+1}+\chi_{n+1}(t)=0,\quad n\geqslant 1.$$

It is easy to see that the functions $\Phi_{n+1}(t)$ and $\chi_{n+1}(t)$ depend only on the expansion coefficients with numbers smaller than $(n+1)$. For example,

$$\Phi_2=-\lambda_1 c_0 T_1^4,\quad \chi_2=\frac{q_2}{c_0}T_1^2,$$

$$-\Phi_3=4\lambda_0 c_0 T_1 T_2^2-8\lambda_1 c_0 T_1^3 T_2-\frac{c_1}{c_0}T_1^2\eta_2,$$

$$c_0\chi_3=2q_2T_1T_2+q_3T_1^3-2c_0T_2(\eta_2+\gamma_0\dot{\eta}_2)-c_1T_1^2\eta_2+$$
$$+(c_0\gamma_1+\gamma_0c_1)(T_1\ddot{T}_2-T_1^2\dot{\eta}_2)+\dot{T}_2(22\alpha_0T_1T_2+7\alpha_0T_1^3-4c_0\gamma_0\eta_2).$$

Eliminating $\eta_{n+1}(t)$ from (9), we write the second order linear differential equation

$$\gamma_0\frac{n+1}{n}\ddot{T}_{n+1}+\left[(2n+3)\gamma_0\theta_1+\frac{n+1}{n}\right]\dot{T}_{n+1}-\theta_1(n+1)^2T_{n+1}+$$
$$+\frac{\gamma_0}{n}\dot{\Phi}_{n+1}+\frac{n+1}{n}\Phi_{n+1}+\chi_{n+1}=0,\quad \gamma_0,\ \theta_1-\text{const},$$

whose general solution has the form

$$T_{n+1}(t)=P_{n+1}\exp(\varepsilon_{n+1}t)+R_{n+1}\exp(\varkappa_{n+1}t)-\left(\int_0^t\frac{f_{n+1}\exp(-\varepsilon_{n+1}t)}{\varkappa_{n+1}-\varepsilon_{n+1}}dt\right)\exp(\varepsilon_{n+1}t)+$$
$$+\left(\int_0^t\frac{f_{n+1}\exp(-\varkappa_{n+1}t)}{\varkappa_{n+1}-\varepsilon_{n+1}}dt\right)\exp(\varkappa_{n+1}t),$$
$$f_{n+1}(t)=\frac{-n}{\gamma_0(n+1)}\left(\frac{\gamma_0}{n}\dot{\Phi}_{n+1}+\frac{n+1}{n}\Phi_{n+1}+\chi_{n+1}\right),\quad\gamma_0\neq 0, \tag{10}$$
$$\varepsilon_{n+1},\ \varkappa_{n+1}=\left\{-\left[(2n+3)\gamma_0\theta_1+\frac{n+1}{n}\right]\pm\left[\left((2n+3)\gamma_0\theta_1+\frac{n+1}{n}\right)^2+\frac{4\gamma_0}{n}(n+1)^3\theta_1\right]^{\frac{1}{2}}\right\}\frac{n}{2\gamma_0(n+1)}.$$

The value of ε_{n+1} is positive; it is calculated by using the positive value of the square root. The value of $\varkappa_{n+1}$ is negative; it is calculated by using the negative value of the square root. For this reason we take

$$P_{n+1}=0,\quad n\geqslant 1,$$

the R_{n+1} are arbitrary integration constants.

Consequently, at time zero we have the following temperature distribution:

$$T(\xi,\ 0)=T_1\xi+\sum_{n=1}^{\infty}R_{n+1}\xi^{n+1},\quad T_1\neq 0,$$

$$x(\xi,\ 0)=\int\frac{d\xi}{S(\xi,\ 0)};\quad \xi\to 0,\ x\to-\infty,\ T\to 0;$$

the constants R_{n+1} are chosen so that the function $T(\xi, 0)$ will be differentiable a sufficient number of times.

Example.

$$T_1 > 0, \ R_{n+1} = 0, \ n \geqslant 1; \ c_1 = 0, \ S = c_0 T,$$
$$\xi(x, 0) = \exp(c_0 T_1 x), \ T(x, 0) = T_1 \exp(c_0 T_1 x), \ x \in (-\infty, x_b(0)].$$

We assume that the boundary of the semiinfinite region is moving: $x_b(t) = x[\xi_b(t), t]$. The choices of $T_b(t) = T(x_b, t)$ and $x_b(t)$ are interconnected; both these boundary functions depend on the form of the preassigned function $\xi = \xi(t)$ and the choice of the arbitrary constants R_{n+1}, $n \geq 1$.

Relations (5), (8), and (10) give an exact formal solution of the problem. The question of the convergence of series (8) in the neighborhood of the characteristic $\xi = 0$ still remains open.

Let us turn to the question of the boundary layer transition type of solution of the generalized heat-transfer equation.

It was found recently that the term $c\gamma T_{tt}$ in the generalized heat-transfer equation is helpful in a methodological connection: Taking account of the finite rate of propagation of heat increases the stability of an inverse heat-conduction boundary value problem [4, 5]. Here we consider the smoothing effect of the $c\gamma T_{tt}$ term for one form of discontinuous initial data.

Suppose we have the parabolic equation (1) and the hyperbolic equation

$$cT_t + c\gamma T_{tt} = (\lambda T_x)_x. \tag{11}$$

Both of these equations are second order in the argument x; we specify identical boundary conditions with respect to the coordinate x for both equations. We do not make the form of the boundary conditions specific, since it is not important for the following arguments.

We assume that the relaxation period γ is short, and discuss the situation relative to the presence of this small parameter for the old time derivative in Eq. (11). In this analysis we use the approach employed in [6] to study the relation between the Navier–Stokes and Euler equations for a gas whose viscosity approaches zero.

The object of our discussion will be the initial conditions for Eqs. (1) and (11). Equation (11) must have two conditions with respect to the argument t, and Eq. (1) one initial condition. For the parabolic Eq. (1) we specify discontinuous initial data: At $t = 0$ the temperature jumps from $T_0(x)$ to $T_1(x)$ in the spatial region under consideration. For the hyperbolic Eq. (11) we pose an initial two-point problem with one of the conditions at infinity:

$$t = 0, \ T = T_0(x); \ t \to \infty, \ T \to T_1(x).$$

We now assume that the relaxation period is constant: $\gamma = \gamma_0 \equiv \text{const}$. Following the Mises–Ladford method [6] we introduce a new argument $s = t/\gamma_0$, and obtain instead of Eq. (11)

$$cT_s + cT_{ss} = \gamma_0 (\lambda T_x)_x. \tag{12}$$

We now replace the right-hand side of Eq. (12) by zero, and consider a solution of this equation for which T approaches finite values as $s \to 0$ and $s \to \infty$; we assume that each limiting temperature depends on x as on a parameter. We denote this solution by $\theta(s, \gamma_0)$, employing a single symbol for the whole family of variable states. We call $\theta(s, \gamma_0)$ a solution of Eq. (12) of the boundary layer transition type corresponding to the time $t = 0$ and the temperatures T_0 and T_1. This term reflects the structure of the solution and the method of obtaining it.

The integration of Eq. (12) with a zero right-hand side gives

$$T = T_1 + (T_0 - T_1)\exp(-s),$$
$$s \to 0, \ T \to T_0; \ s \to +\infty, \ T \to T_1,$$
$$\frac{t}{\gamma_0} = \ln\left|\frac{T_0 - T_1}{T - T_1}\right|.$$

We study the transition from T_0 to T_1. Let δ be any real number such that $0 < \delta < 0.5$; we take two intermediate values of the temperature T' and T" such that

$$T_0 - T' = \delta(T_0 - T_1) = T'' - T_1, \quad [T', T''] \subseteq [T_0, T_1],$$

then

$$T_0 - T'' = (1-\delta)(T_0 - T_1), \quad T' - T'' = (1-2\delta)(T_0 - T_1).$$

Employing these relations we calculate the difference of the corresponding values of the arguments:

$$t'' - t' = \gamma_0 \ln\left(\frac{1}{\delta} - 1\right).$$

We conclude from this result that the time interval during which the main part of the change from T_0 to T_1 occurs approaches zero as $\gamma_0 \to 0$. We note that the right-hand side of the last equation does not depend on the temperature difference $T_0 - T_1$.

Let us assume that a certain heat-transfer problem has been solved with the hyperbolic Eq. (11) in the (x, t) plane for all small $\gamma_0 \neq 0$. We denote this family of solutions by $\tau(x, t; \gamma_0)$. We make the substitution $s = t/\gamma_0$ in these solutions and denote this same family by $\tau(x, s; \gamma_0)$. We assume now that this same boundary value problem is solved by the parabolic Eq. (1); the solution obtained $\tau_0(x, t)$ characterizes the temperature distribution everywhere except at $t = 0$, where unbounded values of T_t may occur.

On the basis of the Mises–Ladford construction it is natural to assume that for sufficiently small γ_0 the solution τ of the generalized heat-transfer equation will be close to τ_0 everywhere except in the neighborhood of $t = 0$, where the temperature experiences a jump. During the jump, $\tau(x, t; \gamma_0)$ has derivatives which increase without bound as $\gamma_0 \to 0$, but it seems plausible that the derivatives $\partial/\partial s = \gamma_0 \partial/\partial t$ of the solution $\tau(x, s; \gamma_0)$ are bounded. Then the right-hand side of Eq. (12) is of the order γ_0, so that τ turns out to be close to the boundary layer transition type of solution $\theta(s, \gamma_0)$ of Eq. (12) in this interval.

Let us analyze these results. Let the time interval $\Delta(\gamma_0)$ approach zero more slowly than γ_0, i.e., $\gamma_0/\Delta(\gamma_0) \to 0$. Then by choosing a sufficiently small value of γ_0 it is possible to satisfy the following requirements with a preassigned accuracy:

1) at times $t > \Delta(\gamma_0)$ the solution $\tau(x, t; \gamma_0)$ is approximately equal to the solution $\tau_0(x, t)$;

2) at times $t < \Delta(\gamma_0)$ the solution $\tau(x, s; \gamma_0)$ is approximately equal to $\theta(s, \gamma_0)$ – the solution of the boundary layer transition type corresponding to $t = 0$ and the temperatures T_0 and T_1.

There has not been a rigorous mathematical investigation of the general case of the transition from the hyperbolic to the parabolic heat-transfer equation as $\gamma \to 0$. Certain questions of the degeneration of the hyperbolic equation with a small parameter into the parabolic equation for rapidly oscillating boundary conditions were studied in [7].

NOTATION

T, temperature; x, Cartesian coordinate; t, time; c, volumetric heat capacity; λ, thermal conductivity of medium; γ, heat-transfer (heat flux) relaxation period; q_v, internal heat source strength; ξ, new argument; η, auxiliary function. Subscripts: an independent variable as a subscript denotes partial differentiation; a dot over a quantity denotes ordinary differentiation.

LITERATURE CITED

1. A. V. Lykov, Heat and Mass Transfer (Handbook) [in Russian], Énergiya, Moscow (1978).
2. P. M. Kolesnikov, Energy Transport in Inhomogeneous Media [in Russian], Nauka i Tekhnika, Minsk (1974).
3. V. A. Bubnov, "Molecular kinetic basis of the heat-transfer equation," Inzh.-Fiz. Zh., 28, 670 (1975).
4. I. A. Novikov, "Hyperbolic heat-conduction equation. Solution of direct and inverse problems for a semi-infinite rod," Inzh.-Fiz. Zh., 35, 734 (1978).
5. O. M. Alifanov, Identification of Heat-Transfer Processes of Aircraft [in Russian], Mashinostroenie, Moscow (1979).
6. R. Von Mises, Mathematical Theory of Compressible Fluid Flow, rev. ed., Academic Press (1958).
7. M. I. Vishik and L. A. Lyusternik, "Asymptotic behavior of solutions of linear differential equations with large or rapidly varying coefficients and boundary conditions," Usp. Mat. Nauk, 15, No. 4, 27 (1960).